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ORIGINAL ARTICLE

Weak and strong convergence theorems of modified Ishikawa iteration for an infinitely countable family of pointwise asymptotically nonexpansive mappings in Hilbert spaces

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Projection technique

Abstract In this paper, we first verify that the sequence generated by the Ishikawa iterative scheme is weakly convergent to a fixed point of a uniformly Lipschitzian and pointwise asymptotically nonexpansive mapping T in a Hilbert space. Then, we introduce a new kind of monotone hybrid method which is a modification of the Ishikawa iterative scheme for finding a common fixed point of an infinitely countable family of uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings in a Hilbert space. We also prove the strongly convergent of the sequence generated by the proposed monotone hybrid method, for an infinitely countable family of uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings in a Hilbert space. The results presented in this paper extend and improve some known results in the literature.

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1. Introduction

Let C be a nonempty subset of a normed space X and let $T : C \to C$ be a self-mapping. We denote as $\mathrm{Fix}(T)$ the set of all fixed points of T, that is $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Recall that the mapping T is said to be

(i) *nonexpansive* if, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$;
(ii) *asymptotically nonexpansive* (Goebel and Kirk, 1972) if, there exists a sequence $\{\gamma_n\}$ in $[1, +\infty)$ with $\lim_{n\to\infty}\gamma_n = 1$ such that $\|T^n x - T^n y\| \leqslant \gamma_n \|x - y\|$, for all $x, y \in C$ and $n \in \mathbb{N}$;
(iii) *uniformly Lipschitzian* if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leqslant L\|x - y\|$, for all $x, y \in C$ and $n \in \mathbb{N}$. Evidently, every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

Construction of fixed points of nonexpansive mappings and asymptotically nonexpansive mappings is an important subject in the theory of nonexpansive mappings and finds its applications in a number of applied areas, in particular in image recovery and signal processing (see, for example, Byrne, 2004; Podilchuk and Mammone, 1990; Sezan and Stark, 1987; Youla, 1987, 1990).

However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology and since averaged iterations prevail. Mann (1953) introduced the following iterative procedure for approximating a fixed point of a nonexpansive mapping T in a Hilbert space $\mathcal{H}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities, see Mann (1953), Opial (1967) and Schu (1991). However Mann iteration processes have only weak convergence even in a Hilbert space, for instance, see Kim and Xu (2006); Mann (1953); Takahashi et al. (2008). Even, Reich (1979) proved that if X is a uniformly convex Banach space with a Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T.

Some attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi (2003) proposed the following modification of the Mann iteration method (1.1) for a single nonexpansive mapping T in a Hilbert space $\mathcal{H}$:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leqslant \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.2}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above by 1, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{\mathrm{Fix}(T)}x_0$.

Subsequently, Mann iteration method (1.1) has been modified for finding a fixed point of asymptotically nonexpansive mapping as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ (see, for example, Cholamjiak and Suantai, 2010; Kim and Xu, 2006; Tan and Xu, 1993). Similarly, we note that the modified Mann's iteration (1.3) has only weak convergence and is in general not strongly convergent for asymptotically nonexpansive mappings. In order to get strong convergence, Kim and Xu (2006) introduced the following modification of (1.3) for finding a fixed point of a single asymptotically nonexpansive mapping T in a Hilbert space $\mathcal{H}$ to extend the result of Nakajo and Takahashi (2003) from a single nonexpansive mapping to a single asymptotically nonexpansive mapping:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam}\, C)^2 \to 0$, as $n \to \infty$. They proved that if $\alpha_n \leqslant a$ for all $n \in \mathbb{N}$ and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{\mathrm{Fix}(T)}x_0$.

We observe that the iterative algorithms (1.2) and (1.4) generate a sequence $\{x_n\}$ by projecting x_0 onto the intersection of the suitably constructed closed convex sets C_n and Q_n. Takahashi et al. (2008) introduced the following modification of the Mann's iteration method (1.1) which just involved one closed convex set for a family of nonexpansive mappings $\{T_n\}$:

$$\begin{cases} u_0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_1 = C, \quad u_1 = P_{C_1} x_0, \\ y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leqslant \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{1.5}$$

They proved that if $\alpha_n \leqslant a$ for all n and for some $0 < a < 1$, then the sequence $\{u_n\}$ generated by (1.5) converges strongly to $P_{\mathrm{Fix}(T)}x_0$.

Recently, Inchan (2008) introduced a hybrid method of modified Mann's iteration (1.3) for an asymptotically nonexpansive mapping T as below:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \quad x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$, as $n \to \infty$. He proved that if $0 \leqslant \alpha_n \leqslant a < 1$ for all n and for some a, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{\mathrm{Fix}(T)}x_0$.

Kirk and Xu (2008) introduced pointwise asymptotically nonexpansive mappings as below.

Definition 1.1. A mapping $T : C \to C$ is called *pointwise asymptotically nonexpansive* if, for each $n \in \mathbb{N}$ and each $x, y \in C$, we have $\|T^n x - T^n y\| \leqslant \alpha_n(x)\|x - y\|$, where $\alpha_n \to 1$ pointwise on C.

It is clear that an asymptotically nonexpansive mapping is pointwise asymptotically nonexpansive. It is not hard to see that if C is bounded then a pointwise asymptotically nonexpansive T is of asymptotically nonexpansive type, that is, T satisfies the following condition.

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \leqslant 0, \quad \forall x \in C.$$

Ishikawa (1974) introduced the following iterative scheme which is a generalization of the Mann's iterative algorithm (1.1):

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tz_n, \quad n \geqslant 0, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate control sequences in $[0, 1]$. However Ishikawa iteration processes has only weak convergence even in a Hilbert space, for instance, see Ishikawa (1974).

Our modified Ishikawa iteration method generates a sequence $\{x_n\}$ recursively via

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n z_n, \quad n \geqslant 0, \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate control sequences in $[0,1]$ and $T : C \to C$ is a pointwise asymptotically nonexpansive mapping with the sequence of mappings $\gamma_n : C \to [1, +\infty)(n \in \mathbb{N})$ satisfying $\lim_{n\to\infty}\gamma_n(x) = 1$, for all $x \in C$.

If $\beta_n = 1$, for all $n \geqslant 0$, then the modified Ishikawa iteration method (1.8) changes into the following modified Mann iteration method:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad n \geqslant 0, \tag{1.9}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and T are the same as in (1.8).

Motivated and inspired by the above works, in this paper, we first establish that the sequence $\{x_n\}$ generated by the Ishikawa iteration scheme (1.8) is weakly convergent to a fixed point of a uniformly Lipschitzian and pointwise asymptotically nonexpansive mapping T in a Hilbert space. Then, we introduce a new type of monotone hybrid method which is a modification of the Ishikawa iteration scheme (1.8) for finding a common fixed point of an infinitely countable family of uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$. We also prove the strong convergence of the sequence generated by the proposed monotone hybrid method, for an infinitely countable family of uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings in a Hilbert space.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space which is equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let C be a nonempty closed convex subset of $\mathcal{H}$. We denote by $d_C(\cdot)$ the usual distance function to C, i.e., $d_C(u) = \inf_{v\in C}\|u - v\|$. Let $u \in \mathcal{H}$ be a point not lying in C. A point $v \in C$ is called a *closest point or a projection* of u onto C if, $d_C(u) = \|u - v\|$, i.e., $v = P_C u$ if and only if $\|u - P_C u\| \leqslant \|u - w\|$, for all $w \in C$. The mapping $P_C : \mathcal{H} \to C$ is called the *metric projection* of $\mathcal{H}$ onto C. We know that P_C is a nonexpansive mapping.

We will use $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. For given sequence $\{x_n\} \subseteq C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak limit set of $\{x_n\}$.

We need some facts and tools in a real Hilbert space $\mathcal{H}$ which are listed as lemmas below.

Lemma 2.1 (Tan and Xu, 1993). *Let $\{a_n\}$ and $\{\delta_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1 + \delta_n)a_n, \quad \forall n = 1, 2, 3, \ldots$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$, then $\lim_{n\to\infty}a_n$ exists.

Lemma 2.2 (Marino and Xu, 2007). *Let $\mathcal{H}$ be a real Hilbert space. Then for each $x, y \in \mathcal{H}$ and each $t \in [0,1]$*

(a) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.
(b) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.
(c) *If* $\{x_n\}$ *is a sequence in* $\mathcal{H}$ *weakly convergent to* z, *then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2.$$

Lemma 2.3 (Marino and Xu, 2007). *Let* C *be a closed convex subset of a real Hilbert space* $\mathcal{H}$ *and let* P_C *be the metric projection from* $\mathcal{H}$ *onto* C. *Given* $x \in \mathcal{H}$ *and* $z \in C$. Then $z = P_C x$ if and only if for each $y \in C$ we have $\langle x - z, y - z\rangle \leqslant 0,\ \forall y \in C$.

Lemma 2.4 (Martinez-Yanes and Xu, 2006). *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *For each* $x, y, z \in \mathcal{H}$ *and* $a \in \mathbb{R}$, *the set*

$$D := \{v \in C : \|y - v\|^2 \leqslant \|x - v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.5 (Martinez-Yanes and Xu, 2006). *Let* C *be a closed convex subset of a real Hilbert space* $\mathcal{H}$ *and* $\{x_n\}$ *be a sequence in* $\mathcal{H}$. *Let* $u \in \mathcal{H}$ *and* $q = P_C u$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subseteq C$ *and satisfies the condition* $\|x_n - u\| \leqslant \|u - q\|$, *for all* $n \in \mathbb{N}$, *then* $x_n \to q$.

Lemma 2.6 (Nakajo and Takahashi, 2003). *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and* $P_C : \mathcal{H} \to C$ *be the metric projection from* $\mathcal{H}$ *onto* C. *Then* $\|y - P_C x\|^2 + \|x - P_C x\|^2 \leqslant \|x - y\|^2$, for all $x \in \mathcal{H}$ and $y \in C$.

Here, we will discuss basic properties of pointwise asymptotically nonexpansive mappings which will be used in the next section.

Proposition 2.7 (Demiclosedness principle). *Let* C *be a closed convex subset of a Hilbert space* $\mathcal{H}$ *and let* $T : C \to C$ *be a uniformly* L*-Lipschitzian and pointwise asymptotically nonexpansive mapping with the sequence of mappings* $\gamma_n : C \to [1, +\infty)$ $(n \in \mathbb{N})$ *satisfying* $\lim_{n\to\infty}\gamma_n(x) = 1$, *for all* $x \in C$. *Then* $I - T$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup q$ *and* $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$, *then* $(I - T)q = 0$.

Proof. Since the sequence $\{x_n\}$ is bounded, we can define a function ϕ on $\mathcal{H}$ by

$$\phi(x) = \limsup_{n\to\infty} \|x_n - x\|^2, \quad x \in \mathcal{H}.$$

From $x_n \rightharpoonup q$ and Lemma 2.2(c), we conclude that

$$\phi(x) = \phi(q) + \|x - q\|^2, \quad \forall x \in \mathcal{H}.$$

In particular, for each $m \in \mathbb{N}$, we have

$$\phi(T^m q) = \phi(q) + \|T^m q - q\|^2. \tag{2.1}$$

On the other hand, pointwise asymptotically nonexpansivity of T implies that

$$\begin{aligned}\phi(T^m q) &= \limsup_{n\to\infty} \|x_n - T^m q\|^2 \\ &= \limsup_{n\to\infty} \|x_n - T^m x_n + T^m x_n - T^m q\|^2 \\ &= \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\langle x_n - T^m x_n, T^m x_n - T^m q\rangle + \|T^m x_n - T^m q\|^2) \\ &\leqslant \limsup_{n\to\infty} \|x_n - T^m x_n\|(\|x_n - T^m x_n\| + 2L\|x_n - q\|) \\ &\quad + \limsup_{n\to\infty} \gamma_m^2(q)\|x_n - q\|^2.\end{aligned}$$

Since $\lim_{m\to\infty}\gamma_m(x) = 1$, for each $x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$, taking $\limsup_{m\to\infty}$ from both sides of the above inequality, we derive that

$$\limsup_{m\to\infty} \phi(T^m q) \leqslant \limsup_{n\to\infty} \|x_n - q\|^2 = \phi(q). \tag{2.2}$$

Combining (2.1) and (2.2), it follows that $\limsup_{m\to\infty}\|q - T^m q\|^2 = 0$, i.e., $T^m q \to q$, hence $Tq = q$. $\square$

Since every asymptotically nonexpansive mapping is a uniformly Lipschitzian mapping, we have the following statement for asymptotically nonexpansive mappings.

Corollary 2.8 (Lin et al., 1995). *Let C be a bounded closed convex subset of a Hilbert space $\mathcal{H}$ and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Corollary 2.9 (Opial, 1967; Goebel and Kirk, 1972). *Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.*

Proposition 2.10. *Let C, $\mathcal{H}$ and T be the same as in Proposition* 2.7. *Then the fixed point set $Fix(T)$ of T is closed and convex so that projection $P_{Fix(T)}$ is well defined.*

Proof. To see that $\mathrm{Fix}(T)$ is closed, suppose $\{p_n\} \subseteq \mathrm{Fix}(T)$ and $p_n \to p$. Then for all $n \in \mathbb{N}$,

$$\|Tp - p\| \leqslant \|Tp - p_n\| + \|p_n - p\| = \|Tp - Tp_n\| + \|p_n - p\|.$$

Since $p_n \to p$ as $n \to \infty$ and T is continuous, the right side of the above inequality approaches to zero as $n \to \infty$, hence $p \in \mathrm{Fix}(T)$ and so $\mathrm{Fix}(T)$ is closed.

To see the convexity of $\mathrm{Fix}(T)$, we need to prove that $\lambda u + (1-\lambda)v \in \mathrm{Fix}(T)$, whenever $u, v \in \mathrm{Fix}(T)$ and $\lambda \in (0,1)$. Set $w = \lambda u + (1-\lambda)v$. We note that $\|u-w\| = (1-\lambda)\|u-v\|$ and $\|v-w\| = \lambda\|u-v\|$. Then by Lemma 2.2(b), we get

$$\begin{aligned}
\|w - T^n w\|^2 &= \|\lambda u + (1-\lambda)v - T^n w\|^2 = \|\lambda(u - T^n w) + (1-\lambda)(v - T^n w)\|^2 \\
&= \lambda\|u - T^n w\|^2 + (1-\lambda)\|v - T^n w\|^2 - \lambda(1-\lambda)\|u-v\|^2 \\
&\leqslant \lambda\gamma_n^2(u)\|u-w\|^2 + (1-\lambda)\gamma_n^2(v)\|v-w\|^2 - \lambda(1-\lambda)\|u-v\|^2 \\
&\leqslant \lambda(1-\lambda)^2\tilde{\gamma}_n^2\|u-v\|^2 + \lambda^2(1-\lambda)\tilde{\gamma}_n^2\|u-v\|^2 - \lambda(1-\lambda)\|u-v\|^2 \\
&= \lambda(1-\lambda)(\tilde{\gamma}_n^2 - 1)\|u-v\|^2, \qquad (2.3)
\end{aligned}$$

where $\tilde{\gamma}_n = \max\{\gamma_n(u), \gamma_n(v)\}$, for each $n \in \mathbb{N}$. By taking the limit in (2.3) as $n \to \infty$ and using the fact that $\tilde{\gamma}_n \to 1$ as $n \to \infty$, we get $T^n w \to w$, hence $Tw = w$, and therefore $\mathrm{Fix}(T)$ is convex. $\square$

3. Weak convergence of the modified Ishikawa iteration method

In this section, we shall prove that the sequence generated by the Ishikawa iteration method (1.8) is weakly convergent to a fixed point of a pointwise asymptotically nonexpansive mapping T and in general is not strongly convergent.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$ and $T : C \to C$ a uniformly L-Lipschitzian and pointwise asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$ and $\gamma_n : C \to [1, +\infty)(n \in \mathbb{N})$ satisfying $\lim_{n\to\infty}\gamma_n(x) = 1$, for all $x \in C$. Suppose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\limsup_{n\to\infty}\alpha_n < 1$, $\limsup_{n\to\infty}\beta_n < 1$ and $\liminf_{n\to\infty}\beta_n > 0$. Moreover, assume that $\sum_{n=1}^{\infty}(\gamma_n^4(p) - 1) < \infty$ for each $p \in Fix(T)$. Then the sequence $\{x_n\}$ generated by the modified Ishikawa iteration method (1.8) converges weakly to a fixed point of T.*

Proof. Pick $p \in \mathrm{Fix}(T)$. We first show that $\lim_{n\to\infty}\|x_n - p\|$ exists. By using Lemma 2.2(b), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T^n z_n - p)\|^2 \\
&\leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T^n z_n - p\|^2 \\
&\leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\gamma_n^2(p)\|z_n - p\|^2 \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(T^n x_n - p)\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T^n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&\leqslant \beta_n\|x_n - p\|^2 + (1-\beta_n)\gamma_n^2(p)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&= \|x_n - p\|^2 + (1-\beta_n)(\gamma_n^2(p) - 1)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2. \qquad (3.2)
\end{aligned}$$

Substituting (3.2) in (3.1) yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant \alpha_n\|x_n-p\|^2+(1-\alpha_n)\gamma_n^2(p)[\|x_n-p\|^2\\&\quad+(1-\beta_n)(\gamma_n^2(p)-1)\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T^nx_n\|^2]\\&=[\alpha_n+(1-\alpha_n)\gamma_n^2(p)+(1-\alpha_n)(1-\beta_n)\gamma_n^2(p)(\gamma_n^2(p)-1)]\|x_n-p\|^2\\&\quad-\beta_n(1-\alpha_n)(1-\beta_n)\gamma_n^2(p)\|x_n-T^nx_n\|^2\\&\leqslant[1+(1-\alpha_n)(\gamma_n^2(p)-1)+(1-\alpha_n)\gamma_n^2(p)(\gamma_n^2(p)-1)]\|x_n-p\|^2\\&\quad-\beta_n(1-\alpha_n)(1-\beta_n)\gamma_n^2(p)\|x_n-T^nx_n\|^2\\&=[1+(1-\alpha_n)(\gamma_n^4(p)-1)]\|x_n-p\|^2\\&\quad-\beta_n(1-\alpha_n)(1-\beta_n)\gamma_n^2(p)\|x_n-T^nx_n\|^2\\&\leqslant[1+(\gamma_n^4(p)-1)]\|x_n-p\|^2.\end{aligned}\tag{3.3}$$

Since $\sum_{n=1}^{\infty}(\gamma_n^4(p)-1)<\infty$, it follows from (3.3) and Lemma 2.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists. This implies that $\{x_n\}$ is bounded. Since $\limsup_{n\to\infty}\alpha_n<1$, $\limsup_{n\to\infty}\beta_n<1$ and $\liminf_{n\to\infty}\beta_n>0$, we can choose $\epsilon>0$ such that $\alpha_n<1-\epsilon$ and $\epsilon<\beta_n<1-\epsilon$, for large enough $n\in\mathbb{N}$. So we can rewrite (3.3) as follows:

$$\begin{aligned}\epsilon^3\|x_n-T^nx_n\|^2&\leqslant\beta_n(1-\alpha_n)(1-\beta_n)\gamma_n^2(p)\|x_n-T^nx_n\|^2\\&\leqslant[1+(1-\alpha_n)(\gamma_n^4(p)-1)]\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\leqslant(1+(\gamma_n^4(p)-1))\|x_n-p\|^2-\|x_{n+1}-p\|^2,\end{aligned}$$

which leads to

$$\|x_n-T^nx_n\|^2\leqslant\frac{1}{\epsilon^3}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\frac{\gamma_n^4(p)-1}{\epsilon^3}\|x_n-p\|^2.\tag{3.4}$$

Since $\gamma_n(p)\to1$ as $n\to\infty$ and $\lim_{n\to\infty}\|x_n-p\|$ exists it follows from (3.4) that

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0.\tag{3.5}$$

Now, we show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. It follows from (1.8) and uniformly L-Lipschitzian of the mapping T that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|\alpha_nx_n+(1-\alpha_n)T^nz_n-x_n\|=(1-\alpha_n)\|T^nz_n-x_n\|\\&\leqslant(1-\alpha_n)(\|T^nz_n-T^nx_n\|+\|T^nx_n-x_n\|)\\&\leqslant(1-\alpha_n)(L\|z_n-x_n\|+\|T^nx_n-x_n\|)\end{aligned}\tag{3.6}$$

and

$$\|z_n-x_n\|=\|\beta_nx_n+(1-\beta_n)T^nx_n-x_n\|=(1-\beta_n)\|T^nx_n-x_n\|.\tag{3.7}$$

Substituting (3.7) in (3.6) and by using (3.5), we gain

$$\begin{aligned}\|x_{n+1}-x_n\|&\leqslant(1-\alpha_n)(L(1-\beta_n)\|T^nx_n-x_n\|+\|T^nx_n-x_n\|)\\&=(1-\alpha_n)(1+L(1-\beta_n))\|T^nx_n-x_n\|\to0,\end{aligned}\tag{3.8}$$

as $n \to \infty$. Since T is uniformly L-Lipschitzian, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n - Tx_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &\quad + \|T^{n+1}x_n - Tx_n\| \\ &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| \\ &\quad + L\|T^n x_n - x_n\|. \end{aligned} \tag{3.9}$$

It follows from (3.5), (3.8) and (3.9) that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since the sequence $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$, for some $q \in C$. Now $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and Proposition 2.7 imply that $Tq = q$, that is, $q \in \mathrm{Fix}(T)$. We next show that $\{x_n\}$ converges weakly to q. For this end, take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q' \in C$. Again, as above, we conclude that $q' \in \mathrm{Fix}(T)$. Finally, we show that $q = q'$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists for every $p \in \mathrm{Fix}(T)$ and since $q, q' \in \mathrm{Fix}(T)$, by Lemma 2.2(c), we get

$$\begin{aligned}\lim_{n\to\infty}\|x_n - q\|^2 &= \lim_{k\to\infty}\|x_{n_k} - q\|^2 = \lim_{k\to\infty}\|x_{n_k} - q'\|^2 + \|q - q'\|^2 \\ &= \lim_{k\to\infty}\|x_{m_k} - q'\|^2 + \|q - q'\|^2 = \lim_{k\to\infty}\|x_{m_k} - q\|^2 + 2\|q - q'\|^2 \\ &= \lim_{n\to\infty}\|x_n - q\|^2 + 2\|q - q'\|^2.\end{aligned}$$

Therefore $q = q'$ and this completes the proof. □

If $\beta_n = 1$, for all $n \geqslant 0$, in similar way to the proof of Theorem 3.1, one can establish the weakly convergent of iterative sequence generated by the Mann iteration method (1.9) and we omit its proof.

Theorem 3.2. *Let C, $\mathcal{H}$, T and the sequence $\{\alpha_n\}$ be the same as in Theorem 3.1. Suppose $\sum_{n=1}^{\infty}(\gamma_n^2(p) - 1) < \infty$ for each $p \in \mathrm{Fix}(T)$. Then the sequence $\{x_n\}$ generated by the modified Mann iteration method (1.9) converges weakly to a fixed point of T.*

Since every asymptotically nonexpansive mapping is uniformly Lipschitzian, by using Theorem 3.1, we obtain the following theorems for asymptotically nonexpansive mappings.

Theorem 3.3. *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $\{\gamma_n\} \subseteq [1, +\infty)$ $(n \in \mathbb{N})$ satisfying $\lim_{n\to\infty}\gamma_n = 1$. Suppose the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\limsup_{n\to\infty}\alpha_n < 1$, $\limsup_{n\to\infty}\beta_n < 1$ and $\liminf_{n\to\infty}\beta_n > 0$. If, $\sum_{n=1}^{\infty}(\gamma_n^4 - 1) < \infty$, then the sequence $\{x_n\}$ generated by the modified Ishikawa iteration method (1.8) converges weakly to a fixed point of T.*

Theorem 3.4. *Suppose that C, $\mathcal{H}$, T and the sequence $\{\alpha_n\}$ are the same as in Theorem* 3.3 *and $\sum_{n=1}^{\infty}(\gamma_n^2 - 1) < \infty$. Then the sequence $\{x_n\}$ generated by the modified Mann iteration method* (1.9) *converges weakly to a fixed point of T.*

4. Algorithms and strong convergence theorems

In view of Theorems 3.1 and 3.3, we note that the modified Ishikawa iteration method (1.8) in general is not strongly convergent for either pointwise asymptotically nonexpansive mappings or asymptotically nonexpansive mappings. So to get strong convergence one has to modify the iteration (1.8). In this section, we introduce some hybrid iterative algorithms which are just involving one closed convex set for pointwise assymptotically nonexpansive mappings and assymptotically nonexpansive mappings in Hilbert spaces. We also prove the strongly convergent of the sequences generated by the proposed monotone hybrid methods in Hilbert spaces.

Algorithm 4.1. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let for each $i \in \mathbb{N}$, $T_i : C \to C$ be a uniformly L_i-Lipschitzian and pointwise asymptotically nonexpansive mapping with $\mathrm{Fix}(T_i) \neq \emptyset$ and $\gamma_{i,n} : C \to [1, +\infty)$ $(n \in \mathbb{N})$ such that $\lim_{n\to\infty}\gamma_{i,n}(x) = 1$, for all $x \in C$. Suppose that $\{\alpha_{i,n}\}_{n=0}^{\infty}$ and $\{\beta_{i,n}\}_{n=0}^{\infty}(i \in \mathbb{N})$ are appropriate control sequences in $(0,1)$ and let $F := \bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i)\neq\emptyset$. Define the sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{i,1} = C, \quad C_1 = \bigcap\limits_{i=1}^{\infty} C_{i,1}, \quad x_1 = P_{C_1}x_0, \\ y_{i,n} = \alpha_{i,n}x_n + (1-\alpha_{i,n})T_i^n z_n, \\ z_n = \beta_{i,n}x_n + (1-\beta_{i,n})T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leqslant \|x_n - z\|^2 \\ \qquad\qquad -\beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap\limits_{i=1}^{\infty} C_{i,n+1}, \quad n \geqslant 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geqslant 0, \end{cases} \tag{4.1}$$

where for each $i \in \mathbb{N}$ and $n \geqslant 0$, $\theta_{i,n} = (1-\alpha_{i,n})(\gamma_{i,n}^4(p) - 1)\nabla_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in F\} < \infty$.

Now, we verify the strongly convergent of the sequence $\{x_n\}$, generated by the hybrid iterative Algorithm 4.1 for a countable family uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings in a Hilbert space.

Theorem 4.2. *Let C, $\mathcal{H}$, T_i, F, $\theta_{i,n}$, ∇_n and the sequences $\{\alpha_{i,n}\}_{n=0}^{\infty}$ and $\{\beta_{i,n}\}_{n=0}^{\infty}$ for $n \geqslant 0$ and $i \in \mathbb{N}$, be the same as in Algorithm* 4.1. *If* $\limsup_{n\to\infty}\alpha_{i,n} < 1$,

$\limsup_{n\to\infty}\beta_{i,n} < 1$ *and* $\liminf_{n\to\infty}\beta_{i,n} > 0$*, for each* $i \in \mathbb{N}$*, then the sequence* $\{x_n\}$ *generated by Algorithm* 4.1 *converges strongly to* $P_F x_0$.

Proof. First, from Proposition 2.10, we note that $\mathrm{Fix}(T_i)$ is a closed convex subset of C, for each $i \in \mathbb{N}$. So F is a nonempty closed convex subset of C. This implies that the projection P_F is well defined. Now, we show that C_n is closed and convex for all $n \geqslant 1$. For this end, we prove by induction on n that for each $i \in \mathbb{N}$, $C_{i,n}$ is closed and convex. For $n = 1$, $C_{i,1} = C$ is closed and convex. Assume that $C_{i,n}$ is closed and convex for some $n \in \mathbb{N}$. It follows from the definition $C_{i,n+1}$ and Lemma 2.3 that $C_{i,n+1}$ is also closed and convex. Hence $C_{i,n}$ is closed and convex for all $n \in \mathbb{N}$. So C_n is closed and convex for all $n \in \mathbb{N}$. Next we show that $F \subseteq C_n$, for each $n \geqslant 1$. By using Lemma 2.2(b), for each $p \in F$, $i \in \mathbb{N}$ and $n \geqslant 1$, we have

$$\begin{aligned}
\|y_{i,n} - p\|^2 &= \|\alpha_{i,n}(x_n - p) + (1-\alpha_{i,n})(T_i^n z_n - p)\|^2 \\
&\leqslant \alpha_{i,n}\|x_n - p\|^2 + (1-\alpha_{i,n})\|T_i^n z_n - p\|^2 \\
&\leqslant \alpha_{i,n}\|x_n - p\|^2 + (1-\alpha_{i,n})\gamma_{i,n}^2(p)\|z_n - p\|^2 \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\|^2 &= \|\beta_{i,n}(x_n - p) + (1-\beta_{i,n})(T_i^n x_n - p)\|^2 \\
&= \beta_{i,n}\|x_n - p\|^2 + (1-\beta_{i,n})\|T_i^n x_n - p\|^2 - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i^n x_n\|^2 \\
&\leqslant \beta_{i,n}\|x_n - p\|^2 + (1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - p\|^2 - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i^n x_n\|^2 \\
&= \|x_n - p\|^2 + (1-\beta_{i,n})(\gamma_{i,n}^2(p) - 1)\|x_n - p\|^2 \\
&\quad - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i^n x_n\|^2. \qquad (4.3)
\end{aligned}$$

Substituting (4.3) in (4.2) yields

$$\begin{aligned}
\|y_{i,n} - p\|^2 &\leqslant \alpha_{i,n}\|x_n - p\|^2 + (1-\alpha_{i,n})\gamma_{i,n}^2(p)[\|x_n - p\|^2 \\
&\quad + (1-\beta_{i,n})(\gamma_{i,n}^2(p) - 1)\|x_n - p\|^2 - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i^n x_n\|^2] \\
&= [\alpha_{i,n} + (1-\alpha_{i,n})\gamma_{i,n}^2(p) + (1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)(\gamma_{i,n}^2(p) - 1)]\|x_n \\
&\quad - p\|^2 - \beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - T_i^n x_n\|^2 \\
&\leqslant [1 + (1-\alpha_{i,n})(\gamma_{i,n}^2(p) - 1) + (1-\alpha_{i,n})\gamma_{i,n}^2(p)(\gamma_{i,n}^2(p) - 1)]\|x_n \\
&\quad - p\|^2 - \beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - T_i^n x_n\|^2 \\
&\leqslant \|x_n - p\|^2 - \beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - T_i^n x_n\|^2 + \theta_{i,n}. \qquad (4.4)
\end{aligned}$$

Therefore, $p \in C_{i,n}$ for each $i \in \mathbb{N}$ and $n \geqslant 1$. This implies that $F \subseteq C_n$ for each $n \geqslant 1$ and so $C_n \neq \emptyset$. Hence, the sequence $\{x_n\}$ is well defined. It follows from $x_n = P_{C_n}x_0$, $C_{n+1} \subseteq C_n$ and $x_{n+1} \in C_n$ that

$$\|x_n - x_0\| \leqslant \|x_{n+1} - x_0\|, \quad \forall n \geqslant 1. \qquad (4.5)$$

Since $F \subseteq C_n$, for each $n \geqslant 1$, one has

$$\|x_n - x_0\| \leqslant \|z - x_0\|, \quad \forall z \in F, \ \forall n \geqslant 1. \tag{4.6}$$

The inequalities (4.5) and (4.6) imply that the sequence $\{x_n - x_0\}$ is bounded and nondecreasing, hence $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Now, we verify that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m}x_0 \in C_m \subseteq C_n$. By Lemma 2.6, we obtain that

$$\|x_m - x_n\|^2 \leqslant \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{4.7}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, (4.7) implies that $\lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a cauchy sequence in C and so $x_n \to z_0 \in C$ as $n \to \infty$. Therefore, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Now, we show that $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, for each $i \in \mathbb{N}$. Let $i, n \in \mathbb{N}$. Since $x_{n+1} \in C_{i,n}$, it follows from the definition of $C_{i,n}$ that

$$\begin{aligned}\|y_{i,n} - x_{n+1}\|^2 &\leqslant \|x_n - x_{n+1}\|^2 - \beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - T_i^nx_n\|^2 + \theta_{i,n} \\ &\leqslant \|x_n - x_{n+1}\|^2 + \theta_{i,n}. \end{aligned} \tag{4.8}$$

From $\theta_{i,n} \to 0$, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, and (4.8) we deduce that $\|y_{n,i} - x_{n+1}\| \to 0$, as $n \to \infty$. On the other hand, we can rewrite (4.8) as below:

$$\begin{aligned}&\beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2(p)\|x_n - T_i^nx_n\|^2 \\ &\quad \leqslant \|x_n - x_{n+1}\|^2 - \|y_{i,n} - x_{n+1}\|^2 + \theta_{i,n}.\end{aligned} \tag{4.9}$$

Since $\limsup_{n\to\infty}\alpha_{i,n} < 1$, $\limsup_{n\to\infty}\beta_{i,n} < 1$ and $\liminf_{n\to\infty}\beta_{i,n} > 0$, for each $i \in \mathbb{N}$, we can choose $\epsilon > 0$ such that for each $i \in \mathbb{N}$, we have $\alpha_{i,n} < 1 - \epsilon$ and $\epsilon < \beta_{i,n} < 1 - \epsilon$, for large enough $n \in \mathbb{N}$. It follows from (4.9) that

$$\|x_n - T_i^nx_n\|^2 \leqslant \frac{1}{\epsilon^3}(\|x_n - x_{n+1}\|^2 - \|y_{i,n} - x_{n+1}\|^2 + \theta_{i,n}) \to 0, \tag{4.10}$$

as $n \to \infty$. From L_i-Lipschitzian of T_i $(i \in \mathbb{N})$, we conclude that

$$\begin{aligned}\|x_n - T_ix_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1}x_{n+1}\| + \|T_i^{n+1}x_{n+1} \\ &\quad - T_i^{n+1}x_n\| + \|T_i^{n+1}x_n - T_ix_n\| \\ &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - T_i^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| \\ &\quad + L\|T_i^nx_n - x_n\|.\end{aligned} \tag{4.11}$$

The inequalities (4.10), (4.11) and the fact that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ imply that $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, for each $i \in \mathbb{N}$. It follows from the boundedness of $\{x_n\}$, Proposition 2.7 and $\lim_{n\to\infty}\|T_ix_n - x_n\| = 0 (i \in \mathbb{N})$, that $\emptyset \neq \omega_w(x_n) \subseteq F(T_i)$, for each $i \in \mathbb{N}$, hence $\emptyset \neq \omega_w(x_n) \subseteq F$. From (4.6) conclude that $\|x_n - x_0\| \leqslant \|u - x_0\|$ for each $n \geqslant 1$, where $u = P_Fx_0$. Now, Lemma 2.5 guarantees that $x_n \to u$, as $n \to \infty$. This completes the proof. □

Remark 4.3. In the weak convergence result of the iteration process (1.8) for pointwise asymptotically nonexpansive mappings T (Theorem 3.1), there is a restriction on the sequence of mappings $\gamma_n : C \to [1, \infty)$ $(n \in \mathbb{N})$ that is the assumption

$$\sum_{n=1}^{\infty}(\gamma_n^4(p) - 1) < \infty, \quad \forall p \in \mathrm{Fix}(T),$$

while in Theorem 4.2 we do not need this assumption.

If $\beta_{i,n} = 1$, for all $i \in \mathbb{N}$ and $n \geqslant 0$, then Algorithm 4.1 reduces to the following algorithm which is involving the modified Mann iteration for a countable family uniformly Lipschitzian and pointwise asymptotically nonexpansive mappings in a Hilbert space.

Algorithm 4.4. Let C, $\mathcal{H}$, T_i and $\{\alpha_{i,n}\}_{n=0}^{\infty}$ $(i \in \mathbb{N})$ be the same as in Algorithm 4.1 and suppose that $F = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in \mathcal{H} \text{ chosen arbitrarily,} \\ C_{i,1} = C, \quad C_1 = \bigcap\limits_{i=1}^{\infty} C_{i,1}, \quad x_1 = P_{C_1}x_0, \\ y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leqslant \|x_n - z\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap\limits_{i=1}^{\infty} C_{i,n+1}, \quad n \geqslant 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geqslant 0, \end{cases}$$

where for each $i \in \mathbb{N}$ and $n \geqslant 0$, $\theta_{i,n} = (1 - \alpha_{i,n})(\gamma_{i,n}^2(p) - 1)\nabla_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$.

Theorem 4.5. *Let C, $\mathcal{H}$, T_i and the sequences $\{\alpha_{i,n}\}_{n=0}^{\infty}$ $(i \in \mathbb{N})$, be the same as in Algorithm 4.4. If $\limsup_{n\to\infty}\alpha_{i,n} < 1$, for each $i \in \mathbb{N}$, then the sequence $\{x_n\}$ generated by Algorithm 4.4 converges strongly to $P_F x_0$.*

If $T_i = T$, and $\alpha_{i,n} = \alpha_n$, for each $i \in \mathbb{N}$ and $n \geqslant 0$, then Algorithm 4.4 reduces to the following modified Mann iteration algorithm involving a pointwise asymptotically nonexpansive mapping.

Algorithm 4.6. Let C, $\mathcal{H}$ be the same as in Algorithm 4.1 and $T : C \to C$ be a pointwise asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $\gamma_n : C \to [1, \infty)$ satisfying $\lim_{n\to\infty}\gamma_n(x) = 1$, for all $x \in C$. Define the sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geqslant 0, \end{cases}$$

where $\theta_n = (1-\alpha_n)(\gamma_n^2(p) - 1)\nabla_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in F\} < \infty$.

Theorem 4.7. *Let $C, \mathcal{H}, T, F$ and the sequence $\{\alpha_n\}_{n=0}^{\infty}$, be the same as in Algorithm* 4.6. *If* $\limsup_{n\to\infty}\alpha_n < 1$, *then the sequence $\{x_n\}$ generated by Algorithm* 4.6 *converges strongly to $P_F x_0$.*

Algorithm 4.8. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T_i : C \to C (i \in \mathbb{N})$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T_i) \neq \emptyset$ and $\{\gamma_{i,n}\}_{n=0}^{\infty} \subseteq [1, +\infty)$ $(i \in \mathbb{N})$ satisfying $\lim_{n\to\infty}\gamma_{i,n} = 1$, for each $i \in \mathbb{N}$. Assume that $\{\alpha_{i,n}\}_{n=0}^{\infty}$ and $\{\beta_{i,n}\}_{n=0}^{\infty} (i \in \mathbb{N})$ are appropriate control sequences in $(0,1)$ and suppose that $F = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{i,1} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{i,1}, \quad x_1 = P_{C_1} x_0, \\ y_{i,n} = \alpha_{i,n} x_n + (1-\alpha_{i,n}) T_i^n z_n, \\ z_n = \beta_{i,n} x_n + (1-\beta_{i,n}) T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leqslant \|x_n - z\|^2 \\ \qquad\qquad -\beta_{i,n}(1-\alpha_{i,n})(1-\beta_{i,n})\gamma_{i,n}^2 \|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, \quad n \geqslant 1, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geqslant 0, \end{cases}$$

where for each $i \in \mathbb{N}$ and $n \geqslant 0$, $\theta_{i,n} = (1-\alpha_{i,n})(\gamma_{n,i}^4 - 1)\nabla_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in F\} < \infty$.

Theorem 4.9. *Suppose $C, \mathcal{H}, T_i, F$ and the sequences $\{\alpha_{i,n}\}_{n=0}^{\infty}$ and $\{\beta_{i,n}\}_{n=0}^{\infty}$ $(i \in \mathbb{N})$ are the same as in Algorithm* 4.8. *If* $\limsup_{n\to\infty}\alpha_{i,n} < 1$, $\limsup_{n\to\infty}\beta_{i,n} < 1$ *and* $\liminf_{n\to\infty}\beta_{i,n} > 0$, *for each $i \in \mathbb{N}$, then the sequence $\{x_n\}$ generated by Algorithm* 4.8 *converges strongly to $P_F x_0$.*

If $\beta_{i,n} = 1$, for all $i \in \mathbb{N}$ and $n \geqslant 0$, then Algorithm 4.8 collapses to the following algorithm which is involving the modified Mann iteration for a countable family uniformly Lipschitzian and asymptotically nonexpansive mappings in a Hilbert space.

Algorithm 4.10. Let C, $\mathcal{H}$, T_i and $\{\alpha_{i,n}\}_{n=0}^{\infty}$ $(i \in \mathbb{N})$ be the same as in Algorithm 4.8 and assume that $F = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as below:

$$\begin{cases} x_0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{i,1} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{i,1}, \quad x_1 = P_{C_1} x_0, \\ y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leqslant \|x_n - z\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, \quad n \geqslant 1, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geqslant 0, \end{cases} \tag{4.12}$$

where for each $i \in \mathbb{N}$ and $n \geqslant 0$, $\theta_{i,n} = (1 - \alpha_{i,n})(\gamma_{i,n}^2 - 1)\nabla_n^2$, $\nabla_n = \sup_{n \in \mathbb{N}}\{\|x_n - z\| : z \in F\} < \infty$.

If $T_i = T$, and $\alpha_{i,n} = \alpha_n$, for each $i \in \mathbb{N}$ and $n \geqslant 0$, then the modified Mann iteration (4.12) reduces to the modified Mann iteration processes (1.6) introduced by Inchan (2008).

Corollary 4.11. [Theorem 3.1, Inchan, 2008]. *Let $\mathcal{H}$ be a Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. Let T be an asymptotically nonexpansive mapping of C into itself such that $Fix(T) \neq \emptyset$. If $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $z_0 = P_{Fix(T)} x_0$.*

If for each $i \in \mathbb{N}$, $T_i = T$ be a nonexpansive mapping and $\alpha_{i,n}$, for each $i \in \mathbb{N}$ and $n \geqslant 0$, be the same as in Algorithm 4.10, then the modified Mann iteration (4.12) reduces to the modified Mann iteration processes (1.5) introduced by Takahashi et al. (2008).

Corollary 4.12. [Takahashi et al. (2008), Theorem 4.1]. *Let $\mathcal{H}$ be a Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. Let T be a nonexpansive mapping of C into itself such that $Fix(T) \neq \emptyset$. If $0 \leqslant \alpha_n \leqslant a < 1$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $z_0 = P_{Fix(T)} x_0$.*

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